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# Asymptotics for the Moments of Singular Distributions

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We determine the asymptotic behavior of the moments of some singular distributions defined on [0, 1], including the Cantor singular function and some special cases of the Riesz–Nagy singular function. We also conduct some numerical studies of the coefficients in the recurrence relation for the orthogonal polynomials generated by the distributions. 

## 1. INTRODUCTION

A real function defined on [0, 1] which is bounded and non-decreasing and whose set of points of increase (called the support of $d\alpha$, or Supp $d\alpha$) is infinite is called a *distribution*. We denote the function by $\alpha(t)$ and define the *moments* of the function by the Riemann–Stieltjes integral,

$$c_n(\alpha) \equiv c_n = \int_0^1 t^n \, d\alpha(t), \qquad n = 0, 1, 2, \ldots . \tag{1.1}$$

The integral

$$\mathscr{L}(f) = \int_0^1 f(t) \, d\alpha(t) \tag{1.2}$$

can be interpreted as a linear functional on the space of real polynomials. Thus (see [8, 4]), $\mathscr{L}$ generates a system of polynomials $\{p_n(t)\}$ orthogonal with respect to this distribution,

$$\int_0^1 p_m(t)\, p_n(t)\, d\alpha(t) = \begin{cases} 0, & m \neq n, \\ h_n \neq 0, & m = n, \end{cases} \quad m, n = 0, 1, 2, \ldots, \tag{1.3}$$

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$p_n(t)$ being a polynomial of exact degree $n$. If we require the coefficient of $t^n$ in $p_n(t)$ to be 1 (such polynomials are called *monic*) the polynomials satisfy the recurrence formula

$$p_{n+1}(t) = (t + B_n)\, p_n(t) - C_n\, p_{n-1}(t), \qquad n = 1, 2, \ldots. \tag{1.4}$$

Important in the study of such polynomials is the Gram determinant,

$$G_n = \begin{vmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ c_n & c_{n+1} & \cdots & c_{2n} \end{vmatrix}, \qquad n = 0, 1, 2, \ldots. \tag{1.5}$$

The formula in [5, Vol. 2, p. 159] yields the explicit construction for $C_n$ in terms of these determinants,

$$C_n = G_n G_{n-2}/G_{n-1}^2, \qquad n = 2, 3, \ldots. \tag{1.6}$$

One of the major problems in the theory of orthogonal polynomials is to determine the asymptotic behavior as $n \to \infty$ of the coefficient $B_n$ and $C_n$, particularly $C_n$. Several results of this type have been given.[1] Perhaps the most complete account of the situation to date is the result due to Rakhmanov, Máté, and Nevai [13], which states that if

(i) $\alpha(t)$ is *strictly* increasing and

(ii) has a derivative that is positive almost everywhere, then

$$\lim_{n \to \infty} C_n = 1/16. \tag{1.7}$$

Recently some interest has been expressed in the case where $\alpha(t)$ is a pathological distribution, for instance, a function which is not absolutely continuous. Interesting examples of such functions are the so-called singular distributions, where $\alpha'(t)$ exists and is zero almost everywhere. It may even be that the support of $d\alpha$ is not denumerable. A number of such functions are known to measure theorists and are found in the standard works, such as [10]. The asymptotic properties of $C_n$, even for the simplest of these functions, are unknown. The Cantor function violates both conditions of the result above.

[1] The problem is even more difficult when the support of $d\alpha$ is unbounded. A major class of problems in this area is the redoutable Freud problem and its variants. There has accumulated by now an enormous literature on this subject, and some prominent names in this effort are Saff, Lubinsky, Magnus, Nevai, and Rakhmanov. An excellent overview of the subject is given in Nevai's article [11]. See also the references given in [12].

It is known that singular distributions, and their corresponding orthogonal polynomials, may be defined for very large classes of Cantor-like sets, called Julia sets. Among those who have done work in this area are Bessis, Geronimo, Moussa, Hardin, and Van Assche; see Refs. [2, 9, 13] and the references given in these papers. In particular, these authors have studied the equilibrium measure associated with the set. But there is a difficulty in relating their work to the situations we study, since the equilibium measure of the Cantor set is not the classical Cantor function. In fact, the Cantor function is a somewhat artificial—though simple and interesting—function to study. J. Geronimo has studied the equilibrium measure of a Julia set generated by a polynomial, whereas in our cases we work directly on the singular distribution—not necessarily the equilibrium distribution—of a compact set.

Functions more exotic than the Cantor-like functions are the Riesz–Nagy distributions discussed in Section 4. They are *strictly* increasing (unlike the Cantor function). Yet their derivatives exist and are zero almost everywhere. These distributions violate only one condition of the above theorem. Certainly they are not the equilibrium measure of the support [0, 1]; see Van Assche [13, p. 24].

One would expect the asymptotic behavior of the $C_n$ for singular distributions to be more obscure than that described by (1.7). The numerics we have generated confirm that expectation.

In this paper we limit our aims somewhat and determine the asymptotic behavior not of $C_n$ but of the moments $c_n$ for these singular functions. The behavior of $C_n$ can conceivably be extracted from the expression (1.6) although, at present, we do not have the techniques available to do this. However, we do provide numerical values for $C_n$ for those functions we discuss and make a conjecture about the behavior of $C_n$ for general singular distributions. At any rate, the asymptotics of the moments themselves, $c_n$, are interesting.

## 2. Formulation of the Results

In this paper we use the notation $f(x, d\alpha)$ for the exponential generating function of the moment sequence $\{c_n\}$,

$$f(x, d\alpha) := \sum_{n=0}^{\infty} \frac{x^n c_n}{n!}. \tag{2.1}$$

Our approach in all cases is to determine closed form expressions for $f(x)$ and to use contour integration and Mellin transform techniques to establish the asymptotic behavior of $c_n$. A good survey of asymptotic analysis based on the Mellin transform is in Wong's book [16]. For the application of the Mellin transform to singular distributions arising from Julia sets, see [2]. Since $f$ is entire, the Darboux method (which presupposes a finite radius of convergence for $f$) cannot be used and, as is typical in such situations, our computations are very involved. We have tried to keep this work organized by presenting our results as a series of lemmas, some of them interesting in their own right. Our major results are the following:

THEOREM 1. *Let $c_n$ be the nth moment of the Cantor function; then*

$$c_n \sim 2^{1/2-(3\ln 2)/(2\ln 3)} n^{-\ln 2/\ln 3} e^{-2H(n)}, \qquad n \to \infty, \tag{2.2}$$

*where*

$$H(x) = \frac{1}{2\pi i} \sum_{m=-\infty}^{\infty}{}' \frac{1}{m} 2^{-1+2m\pi i/\ln 3}(1-2^{2m\pi i/\ln 3}) \times \zeta\left(1-\frac{2m\pi i}{\ln 3}\right) \Gamma\left(1-\frac{2m\pi i}{\ln 3}\right) x^{2m\pi i/\ln 3}. \tag{2.3}$$

THEOREM 2. *If $c_n$ denotes the nth moment of the Riesz–Nagy function (see Section 4) with respect to a constant sequence $\{\tau\}$, then*

$$c_n \sim \left(\frac{2}{1-\tau}\right)^{1/2} \exp\left(\frac{aE'(1)}{\ln 2}\right) e^{-aM(n)} n^{-\ln(1+a)/\ln 2}, \tag{2.4}$$

*where*

$$a = \frac{1+\tau}{1-\tau}, \qquad E'(1) = \int_0^\infty \frac{\ln x}{e^x + a}\, dx,$$
$$M(x) = \frac{1}{2\pi i} \sum_{m=-\infty}^{\infty}{}' \frac{E(1-2m\pi i/\ln 2)}{m} x^{2m\pi i/\ln 2}. \tag{2.5}$$

The $E(s)$ above is defined as

$$E(s) = \int_0^\infty \frac{x^{s-1}}{e^x + a}\, dx. \tag{2.6}$$

Note that both $H(x)$ and $M(x)$ in the above are real functions for $x > 0$ and are periodic in the variable $\log x$.

For many singular distributions it is known that the moments oscillate with $n$ (see [2]). For the above distributions this is also the case, and the oscillation is reflected in the factors involving $H(x)$ and $M(x)$, respectively. We suspect that this is a general feature of singular distributions, although we do not know how to prove it. This behavior provides a striking contrast with what happens for absolutely continuous distributions; the moments are then ultimately monotonic.

A referee has observed that the fact that the moments of the Cantor function drop off like $n^{-\ln 2/\ln 3}$ reflects the fact that the local measure dimension around the point 1 is $\ln 2/\ln 3$, i.e.,

$$\lim_{\rho \to 0} \frac{\ln \mu(B_\rho(1))}{\ln \rho} = \frac{\ln 2}{\ln 3}, \tag{2.7}$$

where $\mu$ is the measure defined by the distribution and $B_\rho(x)$ is the ball centered at $x$, radius $\rho$. See [7] for another definition of dimension.

Intuitively it is clear that only the behavior of the distribution near the point in its support farthest from 0 (1 in our cases) can be relevant to the asymptotic behavior of the moments. One would suspect that in general the drop-off rate of the moments is bounded above by the local measure dimension about this point. The drop-off rate of moments of the Riesz–Nagy singular distribution is like $n^{-\ln(1+a)/\ln 2}$, which can be very small if $a$ is large. Note $\ln(1+a)/\ln 2$ is the local measure dimension of the distribution at 1. This confirms our speculation.

## 3. The Cantor Function

Let $\alpha(t)$ denote the well-known Cantor function[2] defined on $[0, 1]$. Standard facts about this function are that it is continuous, symmetric about $\frac{1}{2}$, satisfies the functional equation

$$\text{(i)} \quad \alpha\left(\frac{t}{3}\right) = \frac{1}{2}\alpha(t), \qquad 0 \leqslant t \leqslant 1; \tag{3.1}$$

$$\text{(ii)} \quad \alpha\left(\frac{t+2}{3}\right) = \frac{1}{2} + \frac{\alpha(t)}{2}, \qquad 0 \leqslant t \leqslant 1; \tag{3.2}$$

and its moments,

$$c_n := \int_0^1 t^n \, d\alpha(t), \tag{3.3}$$

[2] Reference [10] calls this function the *Lebesgue singular function*.

satisfy the recurrence relation,

$$c_n = \frac{1}{2(3^n - 1)} \sum_{j=0}^{n-1} \binom{n}{j} 2^{n-j} c_j, \qquad n = 1, 2, ..., c_0 = 1. \tag{3.4}$$

Finally, the derivative of $\alpha(t)$ exists when $t$ is in the complement of the Cantor set and is zero there. Thus $\alpha'(t) = 0$ almost everywhere. The support of the function is the Cantor set.

LEMMA 3.1. *The exponential generating function for $c_n$ satisfies the functional equation*

$$f(x) = \frac{e^{2x/3} + 1}{2} f\left(\frac{x}{3}\right) \tag{3.5}$$

*and we have the infinite product representation*

$$f(x) = \prod_{k=1}^{\infty} \left(\frac{e^{2x/3^k} + 1}{2}\right). \tag{3.6}$$

*Proof.* The first equation follows by an elementary power series argument, using the recurrence (3.1). The second follows by iterating the functional equation in the usual way. ∎

$f(x)$ is an entire function with zeros at the points

$$P = \left\{\frac{3^k(2m+1)\pi i}{2} \,\middle|\, m \in Z, k \in \mathbf{N}\right\}. \tag{3.7}$$

We have

$$\frac{c_n}{n!} = \frac{1}{2\pi i} \oint_C \frac{f(t)}{t^{n+1}} \, dt, \tag{3.8}$$

$C$ any simple closed contour encircling the origin. We formally compute the critical point of the integrand. Let $g(x) := \ln f(x) - n \ln x$. Using the product representation above we get

$$g'(x) = 2 \sum_{k=1}^{\infty} \frac{1}{1 + e^{-2x/3^k}} \frac{1}{3^k} - \frac{n}{x}. \tag{3.9}$$

The critical points are those satisfying $g'(x) = 0$.

LEMMA 3.2. *There exists a unique positive real root $\rho_n$ of $g'(x) = 0$.*

*Proof.* Consider the function

$$h(x) := 2x \sum_{k=1}^{\infty} \frac{1}{1+e^{-2x/3^k}} \frac{1}{3^k}. \tag{3.10}$$

$h(x)$ is continuous and positive on $(0, \infty)$, and $x/2 \leqslant h(x) \leqslant x$. Further,

$$h'(x) := 2 \sum_{k=1}^{\infty} \frac{1}{1+e^{-2x/3^k}} \frac{1}{3^k} + 4x \sum_{k=1}^{\infty} \frac{e^{-2x/3^k}}{(1+e^{-2x/3^k})^2} \frac{1}{9^k} > 0, \tag{3.11}$$

so $h(x)$ is strictly increasing to infinity as $x \to \infty$. Thus there is a unique positive root of $h(x) = n$. ∎

LEMMA 3.3.

$$\rho_n = n + \frac{\ln 2}{\ln 3}\left(1 + \frac{F(n)}{\ln 2}\right) + o(1),$$

*where*

$$F(x) = \sum_{m=-\infty}^{\infty}{}' \, 2^{2m\pi i/\ln 3}(1 - 2^{2m\pi i/\ln 3}) \times \zeta\left(1 - \frac{2m\pi i}{\ln 3}\right) \Gamma\left(1 - \frac{2m\pi i}{\ln 3}\right) x^{2m\pi i/\ln 3}. \tag{3.12}$$

Here the "prime" signifies that the term corresponding to $m = 0$ is dropped.

*Remark.* $F(x)$ is real for $x > 0$ and is periodic in the variable $\ln x$.

*Proof.* Clearly $\rho_n \to \infty$ as $n \to \infty$. We have

$$2 \sum_{k=1}^{\infty} \frac{1}{1+e^{-2\rho_n/3^k}} \frac{1}{3^k} = \frac{n}{\rho_n}. \tag{3.13}$$

But, by the Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty} 2 \sum_{k=1}^{\infty} \frac{1}{1+e^{-2\rho_n/3^k}} \frac{1}{3^k} = 2 \sum_{k=1}^{\infty} \frac{1}{3^k} = 1, \tag{3.14}$$

and so $\lim_{n\to\infty}(n/\rho_n) = 1$, or, $\rho_n = n + o(n)$. To obtain the second term of $\rho_n$ we have to do more work. In fact, we must interrupt our proof for the following

SUBLEMMA.

$$\sum_{k=1}^{\infty} \frac{e^{-2n/3^k}}{1+e^{-2n/3^k}} \frac{1}{3^k} = \frac{\ln 2}{2n \ln 3}\left(1+\frac{F(n)}{\ln 2}\right) + o\left(\frac{1}{n}\right). \tag{3.15}$$

*Proof.* We remark that the most obvious tool to use here, the Euler–McLaurin summation formula, is not productive because of the difficulty in controlling the error term. Define

$$I(x) = \sum_{k=1}^{\infty} \frac{e^{-2x/3^k}}{1+e^{-2x/3^k}} \frac{1}{3^k}, \qquad x \in (0, \infty). \tag{3.16}$$

Clearly, $I(x)$ satisfies the functional equation

$$3I(3x) = \frac{e^{-2x}}{1+e^{-2x}} + I(x). \tag{3.17}$$

We show that $I(x) = O(\ln x/x)$ as $x \to \infty$. The argument is elementary. The maximum of

$$\frac{e^{-2x/3^k}}{1+e^{-2x/3^k}} \frac{1}{3^k} \tag{3.18}$$

as a continuous function of $k$ occurs at $k = \ln(2x)/\ln 3$. (Here $x$ is considered a large fixed number.) Thus

$$\begin{aligned} I(x) &= \sum_{1 \leqslant k \leqslant [\ln 2x/\ln 3]} + \sum_{k \geqslant [\ln 2x/\ln 3]+1} \\ &= I_1 + I_2. \end{aligned} \tag{3.19}$$

The first sum is less than or equal to the number of terms times the maximum term, so

$$I_1 \leqslant \left[\frac{\ln 2x}{\ln 3}\right] \frac{e^{-2x/3^k}}{1+e^{-2x/3^k}} \frac{1}{3^k}\bigg|_{k=\ln 2x/\ln 3} = O\left(\frac{\ln x}{x}\right), \tag{3.20}$$

while $I_2$ consists of decreasing terms,

$$\begin{aligned} I_2 &\leqslant \int_{[\ln 2x/\ln 3]+1}^{\infty} \frac{e^{-2x/3^k}}{1+e^{-2x/3^k}} \frac{1}{3^k}\, dk + \frac{e^{-2x/3^k}}{1+e^{-2x/3^k}} \frac{1}{3^k}\bigg|_{k=[\ln 2x/\ln 3]+1}, \\ &\leqslant \int_{[\ln 2x/\ln 3]+1}^{\infty} e^{-2x/3^k} \frac{1}{3^k}\, dk + O\left(\frac{1}{x}\right) = O\left(\frac{1}{x}\right). \end{aligned} \tag{3.21}$$

Thus $I(x)=O(\ln x/x)$ and so its Mellin transform exists. We define

$$\hat{I}(s):=\int_0^\infty I(x)\,x^{s-1}\,dx, \qquad 0<\operatorname{Re} s<1, \tag{3.22}$$

the convergence being absolute and uniform on any compact subset of the vertical strip. Thus the transform is analytic in the strip. The functional equation for $I(x)$ in (3.17) gives

$$\begin{aligned} 3\cdot 3^{-s}\hat{I}(s) &= \int_0^\infty \frac{e^{-2x}x^{s-1}}{1+e^{-2x}}\,dx+\hat{I}(s), \\ \hat{I}(s) &= \frac{1}{3^{-s+1}-1}\int_0^\infty \frac{e^{-2x}x^{s-1}}{1+e^{-2x}}\,dx. \end{aligned} \tag{3.23}$$

The integral above may be identified in terms of the Riemann zeta function (see [5, Vol. 1]). We have

$$\hat{I}(s)=\frac{2^{-s}(1-2^{1-s})}{(3^{-s+1}-1)}\,\zeta(s)\,\Gamma(s). \tag{3.24}$$

By the Mellin inversion theorem we have

$$I(x)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{2^{-s}(1-2^{1-s})}{(3^{-s+1}-1)}\,\zeta(s)\,\Gamma(s)\,x^{-s}\,ds, \qquad 0<\sigma<1. \tag{3.25}$$

The integrand above is analytic in the right half plane except for the simple poles at $s=1-2m\pi i/\ln 3$, $m\in\mathbf{Z}$. We consider integration along a rectangle with vertices

$$\left(-i\frac{2(j+\frac{1}{2})\,\pi i}{\ln 3}+\sigma,\ -i\frac{2(j+\frac{1}{2})\,\pi i}{\ln 3}+\sigma',\ i\frac{2(j+\frac{1}{2})\,\pi i}{\ln 3}+\sigma,\ i\frac{2(j+\frac{1}{2})\,\pi i}{\ln 3}+\sigma'\right),$$

where $\sigma'>1$ and $j$ is a large positive integer. If we use the Laurent expansion of $\zeta(s)$ about $s=1$,

$$\zeta(s)=\frac{1}{s-1}+\gamma+O(s-1), \tag{3.26}$$

we see the residue at $s=1$ is $-\ln 2/(2x\ln 3)$ and the residue at $s=1-2m\pi i/\ln 3$ $(m\neq 0)$ is

$$\frac{-1}{2x\ln 3}\,2^{2m\pi i/\ln 3}(1-2^{2m\pi i/\ln 3})\,\zeta\left(1-\frac{2m\pi i}{\ln 3}\right)\Gamma\left(1-\frac{2m\pi i}{\ln 3}\right)x^{2m\pi i/\ln 3}.$$

Two important facts about the special functions occurring in the integrand are

(i) $\zeta(s) = O(t)$ if $\operatorname{Re} s \geqslant \delta$, $\delta > 0$.

(ii) $\Gamma(s) = O(e^{-(\pi/2 - \varepsilon)|t|})$, where $t = \operatorname{Im} s$, $|t| \to \infty$, provided $\operatorname{Re} s > 0$.

For these statements see Watson's book [14]. By passing to the limit as $j \to \infty$ we can justify

$$I(x) = \frac{\ln 2}{2x \ln 3}\left(1 + \frac{F(x)}{\ln 2}\right) + \frac{1}{2\pi i}\int_{\sigma' - i\infty}^{\sigma' + i\infty} \frac{2^{-s}(1 - 2^{1-s})}{(3^{-s+1} - 1)}\, \zeta(s)\, \Gamma(s)\, x^{-s}\, ds. \tag{3.27}$$

Because of the exponential decrease of $\Gamma(s)$ at $\sigma' \pm i\infty$, the integral above is easily shown to be $O(x^{-\sigma'})$. Setting $x = n$ finishes the proof of the sublemma. ∎

We continue the proof of Lemma 3.3. Set $\rho_n = n + d_n$, with $d_n = o(n)$. Putting this into (3.13) gives

$$2n \sum_{k=1}^{\infty} \frac{1}{1 + e^{-2(n + d_n)/3^k}} \frac{1}{3^k} + 2d_n \sum_{k=1}^{\infty} \frac{1}{1 + e^{-2(n + d_n)/3^k}} \frac{1}{3^k} = n. \tag{3.28}$$

We consider the error

$$\begin{aligned}
&\left|\sum_{k=1}^{\infty} \frac{1}{1 + e^{-2(n + d_n)/3^k}} \frac{1}{3^k} - \sum_{k=1}^{\infty} \frac{1}{1 + e^{-2n/3^k}} \frac{1}{3^k}\right| \\
&\quad \leqslant \left|\sum_{k=1}^{\infty} \frac{e^{-2(n + d_n)/3^k} - e^{-2n/3^k}}{(1 + e^{-2(n + d_n)/3^k})(1 + e^{-2n/3^k})} \frac{1}{3^k}\right| \\
&\quad \leqslant \left|\sum_{k=1}^{\infty} \frac{(2d_n/3^k)\, e^{-2(n + \theta d_n)/3^k}}{(1 + e^{-2(n + d_n)/3^k})(1 + e^{-2n/3^k})} \frac{1}{3^k}\right| \\
&\quad \leqslant o(n) \left|\sum_{k=1}^{\infty} \frac{e^{\varepsilon n/3^k}}{1 + e^{2n/3^k}} \frac{1}{9^k}\right|,
\end{aligned} \tag{3.29}$$

where, by the mean value theorem, $|\theta| < 1$, and for some $0 < \varepsilon < 1$. By the same method as that employed in the sublemma, one can show that

$$\sum_{k=1}^{\infty} \frac{e^{\varepsilon n/3^k}}{1 + e^{2n/3^k}} \frac{1}{9^k} = O\left(\frac{1}{n^2}\right). \tag{3.30}$$

Thus the error, that is, the left-hand side of (3.29), is $o(1/n)$. Equation (3.28) becomes

$$2n\left(\sum_{k=1}^{\infty}\frac{1}{1+e^{-2n/3^k}}\frac{1}{3^k}+o\left(\frac{1}{n}\right)\right)$$
$$+2d_n\left(\sum_{k=1}^{\infty}\frac{1}{1+e^{-2n/3^k}}\frac{1}{3^k}+o\left(\frac{1}{n}\right)\right)=n. \tag{3.31}$$

Now

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{1+e^{-2n/3^k}}\frac{1}{3^k}&=\frac{1}{2}+\sum_{k=1}^{\infty}\left(\frac{1}{1+e^{-2n/3^k}}-1\right)\frac{1}{3^k}\\&=\frac{1}{2}-\sum_{k=1}^{\infty}\left(\frac{e^{-2n/3^k}}{1+e^{-2n/3^k}}\right)\frac{1}{3^k}\\&=\frac{1}{2}-\left(\frac{\ln 2}{2n\ln 3}\left(1+\frac{F(n)}{\ln 2}\right)+o\left(\frac{1}{n}\right)\right),\end{aligned} \tag{3.32}$$

by the sublemma. Putting this into (3.31) gives

$$n-\frac{\ln 2}{\ln 3}\left(1+\frac{F(n)}{\ln 2}\right)+o(1)+d_n+o(d_n)=n \tag{3.33}$$

or

$$d_n=\frac{\ln 2}{\ln 3}\left(1+\frac{F(n)}{\ln 2}\right)+o(1), \tag{3.34}$$

as claimed. ∎

We now return to the estimation of the Cauchy integral, (3.8). Although we know the asymptotics of $\rho_n$, we cannot directly apply the result of Evgrafov [6, p. 23, Theorem 5], because the conditions are not satisfied. What is required is a rescaling and a deformation of contour. We have

$$\frac{c_n}{n!}=\frac{1}{2\pi i}\frac{1}{\rho_n^n}\oint_C\frac{f(\rho_n x)}{x^{n+1}}\,dx, \tag{3.35}$$

where $C$ is any simple closed curve encircling the origin.

LEMMA 3.4.

$$\frac{1}{2\pi i}\oint_C\frac{f(\rho_n x)}{x^{n+1}}\,dx=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f(\rho_n(1+iy))}{(1+iy)^{n+1}}\,dy. \tag{3.36}$$

*Proof.* We take $C$ to consist of a left half-circle $C_3$ centered at the origin of radius $R$ and the two horizontal line segments, $C_2$ and $C_4$, joining this to the vertical line $C_1: (1-iR, 1+iR)$, i.e., the usual contour encountered in Laplace transform theory. Thus

$$\frac{1}{2\pi i}\oint_C (*) = \frac{1}{2\pi i}\int_{C_1} (*) + \frac{1}{2\pi i}\int_{C_2} (*) + \frac{1}{2\pi i}\int_{C_3} (*) + \frac{1}{2\pi i}\int_{C_4} (*). \quad (3.37)$$

Now

$$\left|\int_{C_3} \frac{f(\rho_n x)}{x^{n+1}}\,dx\right| \leqslant \int_{\pi/2}^{3\pi/2} \frac{|f(\rho_n Re^{i\theta})|}{R^n}\,d\theta \quad (3.38)$$

and

$$|f(\rho_n Re^{i\theta})| = \left|\prod_{k=1}^{\infty}\left(\frac{e^{2\rho_n Re^{i\theta}/3^k}+1}{2}\right)\right| \leqslant \prod_{k=1}^{\infty}\left(\frac{e^{2\rho_n R\cos\theta/3^k}+1}{2}\right). \quad (3.39)$$

Since $\pi/2 \leqslant \theta \leqslant 3\pi/2$, $-1 \leqslant \cos\theta \leqslant 0$, so $0 \leqslant e^{2\rho_n R\cos\theta/3^k} \leqslant 1$. Thus

$$|f(\rho_n Re^{i\theta})| \leqslant \prod_{k=1}^{\infty}\left(\frac{1+1}{2}\right) = 1. \quad (3.40)$$

This implies that $\int_{C_3}(f(\rho_n x)/x^{n+1})\,dx \to 0$ as $R \to \infty$. Next we have

$$\begin{aligned}\left|\int_{C_2} \frac{f(\rho_n x)}{x^{n+1}}\,dx\right| &= \left|\int_0^1 \frac{f(\rho_n(\xi+iR))}{(\xi+iR)^{n+1}}\,d\xi\right| \\ &\leqslant \left|\int_0^1 \frac{|f(\rho_n(\xi+iR))|}{(\xi^2+R^2)^{(n+1)/2}}\,d\xi\right| \leqslant \int_0^1 \frac{|f(\rho_n(\xi+iR))|}{R^{n+1}}\,d\xi. \end{aligned} \quad (3.41)$$

But

$$\begin{aligned}|f(\rho_n(\xi+iR))| &= \left|\prod_{k=1}^{\infty}\left(\frac{e^{2\rho_n(\xi+iR)/3^k}+1}{2}\right)\right| \\ &\leqslant \prod_{k=1}^{\infty}\left(\frac{e^{2\rho_n\xi/3^k}+1}{2}\right) = f(\rho_n\xi), \end{aligned} \quad (3.42)$$

which is independent of $R$. Thus

$$\left|\int_{C_2} \frac{f(\rho_n x)}{x^{n+1}}\,dx\right| \leqslant \frac{1}{R^{n+1}}\int_0^1 f(\rho_n\xi)\,d\xi, \quad (3.43)$$

and this approaches 0 as $R \to \infty$. $\int_{C_4}(*)$ can be handled in the same way.

Taking the limit as $R \to \infty$ gives

$$\frac{1}{2\pi i}\oint_C \frac{f(\rho_n x)}{x^{n+1}}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{f(\rho_n(1+iy))}{(1+iy)^{n+1}}\,dy. \quad \blacksquare \tag{3.44}$$

We now write

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} &\frac{f(\rho_n(1+iy))}{(1+iy)^{n+1}}\,dy \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{\ln f(\rho_n(1+iy)) - n\ln(1+iy)\}\,\frac{dy}{1+iy}.\end{aligned} \tag{3.45}$$

Define

$$J_n(y) = \ln f(\rho_n(1+iy)) = \sum_{k=1}^{\infty} \ln\left(\frac{e^{2\rho_n(1+iy)/3^k}+1}{2}\right). \tag{3.46}$$

We split the integral into two parts,

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{f(\rho_n(1+iy))}{(1+iy)^{n+1}}\,dy &= \frac{1}{2\pi}\int_{|y|\leqslant n^{-\varepsilon}} \exp\{J_n(y) - n\ln(1+iy)\}\,\frac{dy}{1+iy} \\ &\quad + \frac{1}{2\pi}\int_{|y|\geqslant n^{-\varepsilon}} \frac{f(\rho_n(1+iy))}{(1+iy)^{n+1}}\,dy, \\ &= S_1 + S_2.\end{aligned} \tag{3.47}$$

where $\frac{1}{3} < \varepsilon < \frac{1}{2}$.

In turns out that the major contribution comes from $S_1$. $S_2$ is negligible in comparison. We first study the asymptotics of $S_1$.

LEMMA 3.5. (i) *The function $J_n(y)$ is analytic in the unit disk $|y| < 1$;*

(ii) *the Taylor expansion of $J_n(y)$ at $y = 0$ is*

$$J_n(y) = \ln f(\rho_n) + 2i\rho_n\, y\left(\sum_{k=1}^{\infty} \frac{1}{3^k(1+e^{-2\rho_n/3^k})}\right) + \sum_{j=2}^{\infty} a_j(n)\, y^j, \tag{3.48}$$

*where the coefficients $a_j(n)$ have for $j \geqslant 2$ the behavior*

$$a_j(n) = O(2^j), \qquad n \to \infty, \tag{3.49}$$

*the implicit constant being independent of $j$ and $n$.*

*Proof.* (i) As previously remarked, the zero set of $f(x)$ is $\{3^k(2m+1)\pi i/2 \mid m \in Z,\ k \in \mathbf{N}\}$. Thus the zero set of $f(\rho_n(1+iy))$ is $\{3^k(2m+1)\pi/2\rho_n + i \mid m \in Z,\ k \in \mathbf{N}\}$. We conclude that $J_n(y)$ is analytic in the unit disk.

(ii) By Cauchy's theorem,

$$a_j(n) = \frac{1}{2\pi i} \oint_C \frac{J_n(y)}{y^{j+1}}\, dy, \tag{3.50}$$

where $C$ is a simple closed contour contained in the unit disk which encircles the origin.

Doing a two-fold integration by parts gives

$$a_j(n) = \frac{-2}{\pi i j(j-1)} \oint_C y^{-j+1} \left( \sum_{k=1}^{\infty} \frac{\rho_n^2 e^{-2\rho_n(1+iy)/3^k}}{(1+e^{-2\rho_n(1+iy)/3^k})^2} \frac{1}{9^k} \right) dy. \tag{3.51}$$

To estimate $a_j(n)$, we choose the contour $C = \{y \mid |y| = \frac{1}{2}\}$. We first estimate the quantity within the parenthesis in the second integral above. The method is essentially that used to establish the sublemma. We define

$$K(x) := \sum_{k=1}^{\infty} \frac{e^{-2x/3^k}}{(1+e^{-2x/3^k})^2} \frac{1}{9^k}. \tag{3.52}$$

First, take $x$ to be a positive variable. Then

$$K(x) \leqslant \sum_{k=1}^{\infty} \frac{e^{-2x/3^k}}{(1+e^{-2x/3^k})} \frac{1}{3^k} = I(x). \tag{3.53}$$

We already know that $I(x) = O(\ln x/x)$ as $x \to \infty$. Thus $K(x) = O(\ln x/x)$. This estimate guarantees the existence of the Mellin transform of $K(x)$, at least for $0 < \operatorname{Re} s < 1$. It is easily verified that $K(x)$ satisfies the functional equation

$$9K(3x) = \frac{e^{-2x}}{(1+e^{-2x})^2} + K(x). \tag{3.54}$$

Define

$$\hat{K}(s) = \int_0^{\infty} K(x)\, x^{s-1}\, dx, \qquad 0 < \operatorname{Re} s < 1. \tag{3.55}$$

Then from (3.54) we have

$$\hat{K}(s)=\frac{1}{3^{2-s}-1}\int_0^{\infty}\frac{e^{-2x}x^{s-1}}{(1+e^{-2x})^2}\,dx. \tag{3.56}$$

Integration by parts in the integral formula for the Riemann zeta function gives

$$(1-2^{1-s})\,\zeta(s)=\frac{1}{\Gamma(s+1)}\int_0^{\infty}\frac{e^x x^s}{(e^x+1)^2}\,dx, \tag{3.57}$$

and identifying this with (3.56) shows

$$\hat{K}(s)=\frac{2^{-s}(1-2^{2-s})}{3^{2-s}-1}\,(s-1)\,\zeta(s-1)\,\Gamma(s-1). \tag{3.58}$$

Inverting the Mellin transform gives

$$K(x)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{2^{-s}(1-2^{2-s})}{3^{2-s}-1}\,\zeta(s-1)\,\Gamma(s)\,x^{-s}\,ds, \qquad 0<\sigma<1. \tag{3.59}$$

Clearly, the integrand is analytic in the right half plane $\operatorname{Re} s>0$ except for simple poles at $s=2-2m\pi i/\ln 3$, $m\in\mathbf{Z}$. If we translate the contour to the right, say, making it a vertical line at 3, we correct by evaluating residues. The entire procedure can be justified by a method similar to that used in the sublemma. The result is

$$K(x)=\frac{\ln 2}{4x^2\ln 3}\left(1+\frac{G(x)}{\ln 2}\right)+\tilde{K}(x), \tag{3.60}$$

where

$$\tilde{K}(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2^{-3-it}(1-2^{-1-it})(2+it)}{3^{-1-it}-1}\,\zeta(2+it)\,\Gamma(2+it)\,x^{-3-it}\,dt$$

and

$$G(x)=\sum_{m=-\infty}^{\infty}{}'\; 2^{2m\pi i/\ln 3}(1-2^{2m\pi i/\ln 3})\,\zeta\left(1-\frac{2m\pi i}{\ln 3}\right)\Gamma\left(2-\frac{2m\pi i}{\ln 3}\right)x^{2m\pi i/\ln 3}.$$

Two simple facts about $G(x)$ are

(i) $G(x)$ is analytic in $\operatorname{Re} x>0$.

(ii) $G(x)$ is bounded on $x>0$.

Now write $x=\alpha+i\beta$. By analytic continuation (3.60) is valid in $\operatorname{Re} x>0$. We let $x=\rho_n(1+iy)$ as in (3.51) with $y$ on the contour $C=\{y \mid |y|=\frac{1}{2}\}$. Then

$$|\Gamma(2+it)(\rho_n(1+iy))^{-3-it}|=\rho_n^{-3}\,|\Gamma(2+it)|\,|e^{-3-it\ln(1+iy)}|. \quad (3.61)$$

Since $y\in C$, $|\arg(1+iy)|\leqslant \pi/6$. Thus

$$\begin{aligned}|\Gamma(2+it)(\rho_n(1+iy))^{-3-it}| &= \rho_n^{-3}O(e^{-(\pi-\varepsilon)|t|/2})\cdot O(e^{t\arg(1+iy)})\\ &= O(\rho_n^{-3}e^{-(\pi/3-\varepsilon)|t|}),\end{aligned} \quad (3.62)$$

where the order estimate is uniformly good for all $y$. Thus

$$|\tilde{K}(\rho_n(1+iy))|=O(\rho_n^{-3}). \quad (3.63)$$

Putting (3.63) back in (3.60) gives

$$K(\rho_n(1+iy))=\frac{\ln 2}{4\ln 3}\rho_n^{-2}(1+iy)^{-2}\left(1+\frac{G(\rho_n(1+iy))}{\ln 2}\right)+O(\rho_n^{-3}). \quad (3.64)$$

Note that from the definition in (3.60) it is not hard to see that $|G(\rho_n(1+iy))|\leqslant c\ \forall y$ with $|y|=\frac{1}{2}$. Hence $|K(\rho_n(1+iy))|=O(\rho_n^{-2})$, where the big $O$ constant holds uniformly for $y\in\{y \mid |y|=\frac{1}{2}\}$. Using (3.64) to rewrite (3.51) gives

$$\begin{aligned}a_j(n) &= \frac{-2}{\pi i j(j-1)}\oint_C y^{-j+1}K(\rho_n(1+iy))\,\rho_n^2\,dy,\\ &= O(2^j). \quad \blacksquare\end{aligned} \quad (3.65)$$

The Taylor expansion of $J_n(y)-n\ln(1+iy)$ at $y=0$ is

$$\begin{aligned}J_n(y)-n\ln(1+iy) &= \ln f(\rho_n)+\left(a_2(n)-\frac{n}{2}\right)y^2+\sum_{j=3}^{\infty}a_j(n)\,y^j\\ &\quad -n\sum_{k=3}^{\infty}\frac{(-1)^{k-1}\,i^k}{k}\,y^k.\end{aligned} \quad (3.66)$$

We use Lemma 3.5.

$$\begin{aligned}|y|\leqslant n^{-\varepsilon} &\Rightarrow J_n(y)-n\ln(1+iy)\\ &= \ln f(\rho_n)+\left(a_2(n)-\frac{n}{2}\right)y^2+O(n^{-3\varepsilon})+O(n^{1-3\varepsilon})\\ &= \ln f(\rho_n)+\left(a_2(n)-\frac{n}{2}\right)y^2+O(n^{1-3\varepsilon}).\end{aligned} \quad (3.67)$$

Thus

$$
\begin{aligned}
S_1 &= \frac{1}{2\pi}\int_{-n^{-\varepsilon}}^{n^{-\varepsilon}} \exp\left\{\ln f(\rho_n) + \left(a_2(n) - \frac{n}{2}\right) y^2 + O(n^{1-3\varepsilon})\right\} dy \\
&= \frac{1}{2\pi}\int_{-n^{-\varepsilon}}^{n^{-\varepsilon}} \exp\left\{\ln f(\rho_n) + \left(a_2(n) - \frac{n}{2}\right) y^2\right\} (1 + O(n^{1-3\varepsilon}))\, dy \\
&= \frac{f(\rho_n)}{2\pi}\int_{-n^{-\varepsilon}}^{n^{-\varepsilon}} \exp\left\{\left(a_2(n) - \frac{n}{2}\right) y^2\right\} dy (1 + O(n^{1-3\varepsilon})) \\
&= \frac{f(\rho_n)}{2\pi}\left(\frac{1}{\sqrt{n/2 - a_2(n)}}\int_{-n^{-\varepsilon+1/2}}^{n^{-\varepsilon+1/2}} e^{-y^2}\, dy\right) (1 + O(n^{1-3\varepsilon})) \\
&= \frac{f(\rho_n)}{2\pi}\left(\frac{1}{\sqrt{n/2 - a_2(n)}}\int_{-\infty}^{\infty} e^{-y^2}\, dy \right. \\
&\qquad \left. - \frac{1}{\sqrt{n/2 - a_2(n)}}\int_{|y| \geqslant n^{-\varepsilon+1/2}} e^{-y^2}\, dy\right) (1 + O(n^{1-3\varepsilon})) \\
&\sim \frac{f(\rho_n)}{\sqrt{2\pi n}},
\end{aligned}
\tag{3.68}
$$

the last step following because $\frac{1}{3} < \varepsilon < \frac{1}{2}$. We have proved

LEMMA 3.6.

$$S_1 \sim \frac{f(\rho_n)}{\sqrt{2\pi n}}. \tag{3.69}$$

We next establish

LEMMA 3.7.

$$f(\rho_n) \sim 2^{1/2 - (3 \ln 2)/(2 \ln 3)} e^{\ln 2/\ln 3} e^n n^{-\ln 2/\ln 3} e^{F(n)/\ln 3 - 2H(n)}, \tag{3.70}$$

*where*

$$
\begin{aligned}
H(x) = \frac{1}{2\pi i} \sum_{m=-\infty}^{\infty}{}' \frac{1}{m} 2^{-1 + 2m\pi i/\ln 3}(1 - 2^{2m\pi i/\ln 3}) \\
\times \zeta\left(1 - \frac{2m\pi i}{\ln 3}\right) \Gamma\left(1 - \frac{2m\pi i}{\ln 3}\right) x^{2m\pi i/\ln 3},
\end{aligned}
$$

*and $F(n)$ is defined in Lemma* 3.3.

*Remark.* $H(x)$ is real for $x>0$ and is periodic in the variable $\ln x$.

*Proof.* Consider

$$f(x)=\prod_{k=1}^{\infty}\left(\frac{e^{2x/3^k}+1}{2}\right).$$

$$\frac{d}{dx}\ln f(x)=2\sum_{k=1}^{\infty}\frac{1}{1+e^{-2x/3^k}}\frac{1}{3^k}$$

$$=1-2\sum_{k=1}^{\infty}\frac{e^{-2x/3^k}}{1+e^{-2x/3^k}}\frac{1}{3^k}. \tag{3.71}$$

The last summation is just the $I(x)$ which appeared in (3.16). Thus, according to (3.25), we have

$$\frac{d}{dx}\ln f(x)=1-\frac{1}{\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{2^{-s}(1-2^{1-s})}{(3^{-s+1}-1)}\zeta(s)\,\Gamma(s)\,x^{-s}\,ds, \qquad 0<\sigma<1. \tag{3.72}$$

Now $\zeta(s)=O(|t|)$ if $0<\operatorname{Re}s<1$ and $\Gamma(s)$ is still $O(e^{-(\pi-\varepsilon)|t|/2})$. Pick an $a$ such that $0<a<1$. We integrate both sides of (3.72) from $a$ to $x$ to obtain

$$\ln f(x)-\ln f(a)=x-a-\frac{1}{\pi i}\int_a^x d\xi\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{2^{-s}(1-2^{1-s})}{(3^{-s+1}-1)}\zeta(s)\,\Gamma(s)\,\xi^{-s}\,ds$$

$$=x-a-\frac{1}{\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{2^{-s}(1-2^{1-s})}{(3^{-s+1}-1)}\zeta(s)\,\Gamma(s)\frac{x^{-s+1}-a^{-s+1}}{-s+1}\,ds$$

$$=x-a-\frac{1}{\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{2^{-s}(1-2^{1-s})}{(3^{-s+1}-1)}\zeta(s)\,\Gamma(s)\frac{x^{-s+1}}{-s+1}\,ds$$

$$+\frac{1}{\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{2^{-s}(1-2^{1-s})}{(3^{-s+1}-1)}\zeta(s)\,\Gamma(s)\frac{a^{-s+1}}{-s+1}\,ds. \tag{3.73}$$

We now let $a\to 0^+$. The integral on the right immediately above goes to 0 because $0<\sigma<1$. Note also that $f(0)=1$. Hence in the limit we have

$$\ln f(x)=x-\frac{1}{\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{2^{-s}(1-2^{1-s})}{(3^{-s+1}-1)}\zeta(s)\,\Gamma(s)\frac{x^{-s+1}}{1-s}\,ds. \tag{3.74}$$

The integrand has a pole of order 2 at $s=1$ and a simple pole at $s=1-2m\pi i/\ln 3$, for each nonzero $m\in\mathbf{Z}$. We translate the contour to the right of 1 and compensate by selecting the residues at the poles. An elementary but lengthy computation gives

$$\operatorname{Res}(s=1)=\frac{-\ln 2}{2\ln 3}\ln x+\frac{\ln 2\ln 3-3\ln^2 2}{4\ln 3} \tag{3.75}$$

and

$$\operatorname{Res}\left(s=1-\frac{2m\pi i}{\ln 3}\right)=\frac{-1}{2m\pi i}2^{-1+2m\pi i/\ln 3}(1-2^{2m\pi i/\ln 3})$$
$$\times\zeta\left(1-\frac{2m\pi i}{\ln 3}\right)\Gamma\left(1-\frac{2m\pi i}{\ln 3}\right)x^{2m\pi i/\ln 3}.$$

Consequently,

$$\ln f(x)=x-2\left(\frac{\ln 2}{2\ln 3}\ln x-\frac{\ln 2\ln 3-3\ln^2 2}{4\ln 3}\right.$$
$$\left.+H(x)+\frac{1}{2\pi i}\int_{\sigma'-i\infty}^{\sigma'+i\infty}(*)\,ds\right), \tag{3.76}$$

and the integral above is $O(x^{1-\sigma'})$, by arguments similar to those used previously. Substituting $\rho_n$ for $x$ and using Lemma 3.3 plus doing a little algebra give

$$f(\rho_n)\sim 2^{1/2-(3\ln 2)/(2\ln 3)}e^{\ln 2/\ln 3}e^n n^{-\ln 2/\ln 3}e^{F(n)/\ln 3-2H(\rho_n)}.$$

Now by the mean value theorem we have $H(\rho_n)-H(n)=(\rho_n-n)\,H'(\xi_n)$, where $\xi_n$ is between $\rho_n$ and $n$. It is easy to see that $|H'(\xi_n)|=O(1/\xi_n)$. Hence by Lemma 3.3 we have $H(\rho_n)-H(n)=O(1/n)$ and the lemma follows. ∎

We now turn to $S_2$. Note that

$$|f(\rho_n(1+iy))|=\left|\prod_{k=1}^{\infty}\frac{e^{2\rho_n(1+iy)/3^k}+1}{2}\right|\leqslant\prod_{k=1}^{\infty}\frac{e^{2\rho_n/3^k}+1}{2}=f(\rho_n). \tag{3.77}$$

Putting the inequality into the integral definition of $S_2$ gives

$$|S_2|\leqslant\frac{f(\rho_n)}{\pi}\int_{n^{-\varepsilon}}^{\infty}\frac{dy}{(1+y^2)^{(n+1)/2}}=\frac{f(\rho_n)}{\pi}\int_{n^{-\varepsilon}}^{\infty}\frac{dy}{(1+y^2)^{(n-1)/2}(1+y^2)}$$
$$\leqslant\frac{f(\rho_n)}{\pi}\frac{1}{(1+n^{-2\varepsilon})^{(n-1)/2}}\int_0^{\infty}\frac{dy}{(1+y^2)}. \tag{3.78}$$

Now for $n$ sufficiently large

$$(1+n^{-2\varepsilon})^{(n-1)/2}=\exp\left(\frac{n-1}{2}\ln(1+n^{-2\varepsilon})\right)\geqslant\exp\left(\frac{1}{3}n^{1-2\varepsilon}\right). \tag{3.79}$$

Our original choice of $\varepsilon$ in (3.47) implies $1-2\varepsilon>0$. Thus

$$|S_2| = O(f(\rho_n)\, e^{-n^{1-2\varepsilon}/3})$$
$$= o(S_1). \tag{3.80}$$

Assembling everything from Lemmas 3.5, 3.6, and 3.7 and using the fact that

$$\rho_n^n = n^n \left(1 + \frac{\ln 2}{n \ln 3}\left(1 + \frac{F(n)}{\ln 2}\right) + o\left(\frac{1}{n}\right)\right)^n$$
$$\sim n^n e^{(\ln 2/\ln 3)(1 + F(n)/\ln 2)}, \tag{3.81}$$

and employing Stirling's formula give our final result.

THEOREM 1.

$$c_n \sim 2^{1/2 - (3 \ln 2)/(2 \ln 3)} n^{-\ln 2/\ln 3} e^{-2H(n)}, \qquad n \to \infty. \quad \blacksquare \tag{3.82}$$

## 4. THE RIESZ–NAGY FUNCTIONS

The Riesz–Nagy functions are a class of functions indexed by a real sequence $\tau = \{\tau_n\}$, $0 < \tau_n < 1$, $n = 0, 1, 2, \ldots$; see [10, p. 278]. Since many are unfamiliar with this class, we give the construction.

We define inductively a sequence of continuous functions $F_n(\tau, t) \equiv F_n(t)$ as follows. Let $F_0(t) = t$, define $F_1(0) = 0$, $F_1(1) = 1$, $F_1(\frac{1}{2}) = ((1 - t_1)/2) \cdot 0 + ((1 + t_1)/2) \cdot 1$, and define $F_1$ to be linear on $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$.

Suppose that $F_0, F_1, \ldots, F_n$ have been defined. We then define

$$F_{n+1}\left(\frac{k}{2^n}\right) = F_n\left(\frac{k}{2^n}\right), \qquad k = 0, 1, \ldots, 2^n;$$

$$F_{n+1}\left(\frac{2k+1}{2^{n+1}}\right) = \frac{1-\tau_{n+1}}{2} F_{n+1}\left(\frac{k}{2^n}\right) + \frac{1+\tau_{n+1}}{2} F_{n+1}\left(\frac{k+1}{2^n}\right), \qquad k = 0, 1, \ldots, 2^n - 1, \tag{4.1}$$

$$F_{n+1} \text{ linear in the intervals } \left[\frac{k}{2^{n+1}}, \frac{k+1}{2^{n+1}}\right], \qquad k = 0, 1, \ldots, 2^{n+1} - 1.$$

Finally, we define the distribution $F(\tau, t)$ by

$$F(\tau, t) \equiv F(t) := \lim_{n \to \infty} F_n(t). \tag{4.2}$$

$F$ is continuous, and *strictly* increasing, and $F'(t)=0$ almost everywhere. Thus $\operatorname{Supp} dF=[0,1]$ and so $F$ violates only the second condition of Rakhmanov's theorem. The points where $F$ fails to be differentiable are countable, namely, the set $\mathscr{D}$ of dyadic rationals

$$\mathscr{D} := \left\{ t \in [0,1] \,\middle|\, t = \frac{m}{2^n},\ m \text{ odd} \right\}. \tag{4.3}$$

We discuss only the simple case $\tau_n = \tau$, a constant sequence.

LEMMA 4.1. *The function $F(\tau, t) \equiv F(t)$ satisfies the functional equations*

$$\text{(i)}\quad \frac{1-\tau}{1+\tau} F(t) + F\left(\frac{1}{2}\right) = F\left(t + \frac{1}{2}\right), \qquad t \in \left[0, \frac{1}{2}\right], \tag{4.4}$$

$$\text{(ii)}\quad F\left(\frac{t}{2}\right) = \frac{1+\tau}{2} F(t), \qquad t \in [0,1]. \tag{4.5}$$

*Proof.* We show that these equations are satisfied for all members of $\mathscr{D}$. Since $\mathscr{D}$ is dense in $[0,1]$ and $F$ is continuous, the lemma will follow immediately.

We employ induction on $n$, the power of 2 in the denominator of the dyadic rational i.e., the *order* of the dyadic rational. By the construction of $F$ we have

$$F\left(\frac{k}{2^n}\right) = F_n\left(\frac{k}{2^n}\right) = F_{n+1}\left(\frac{k}{2^n}\right) = \cdots = F_m\left(\frac{k}{2^n}\right), \qquad m \geqslant n. \tag{4.6}$$

For $n=1,2$, the truth of (4.4), (4.5) is easily verified. Now assume they are true for all dyadic rationals of order $\leqslant n$. Consider an arbitrary dyadic rational $(2k+1)/2^{n+1} \in [0, \frac{1}{2}]$. Then

$$\begin{aligned}
F\left(\frac{2k+1}{2^{n+1}} + \frac{1}{2}\right) &= F\left(\frac{2(2^{n-1}+k)+1}{2^{n+1}}\right) = F_{n+1}\left(\frac{2(2^{n-1}+k)+1}{2^{n+1}}\right) \\
&= \frac{1-\tau}{2} F_n\left(\frac{2^{n-1}+k}{2^n}\right) + \frac{1+\tau}{2} F_n\left(\frac{2^{n-1}+k+1}{2^n}\right) \\
&= \frac{1-\tau}{2} F\left(\frac{2^{n-1}+k}{2^n}\right) + \frac{1+\tau}{2} F\left(\frac{2^{n-1}+k+1}{2^n}\right) \\
&= \frac{1-\tau}{2} F\left(\frac{1}{2} + \frac{k}{2^n}\right) + \frac{1+\tau}{2} F\left(\frac{1}{2} + \frac{k+1}{2^n}\right) \\
&\qquad \left(\text{note } \frac{k}{2^n} \text{ and } \frac{k+1}{2^n} \in \left[0, \frac{1}{2}\right]\right)
\end{aligned}$$

$$
\begin{aligned}
&=\frac{1-\tau}{2}\left(\frac{1-\tau}{1+\tau}F\left(\frac{k}{2^n}\right)+F\left(\frac{1}{2}\right)\right)\\
&\quad+\frac{1+\tau}{2}\left(\frac{1-\tau}{1+\tau}F\left(\frac{k+1}{2^n}\right)+F\left(\frac{1}{2}\right)\right)\\
&\qquad\text{(by the induction hypothesis)}\\
&=\frac{(1-\tau)^2}{2(1+\tau)}F\left(\frac{k}{2^n}\right)+\frac{1-\tau}{2}F\left(\frac{k+1}{2^n}\right)+F\left(\frac{1}{2}\right)\\
&=\frac{(1-\tau)^2}{2(1+\tau)}F_n\left(\frac{k}{2^n}\right)+\frac{1-\tau}{2}F_n\left(\frac{k+1}{2^n}\right)+F\left(\frac{1}{2}\right)\\
&=\frac{1-\tau}{1+\tau}\left(\frac{(1-\tau)}{2}F_n\left(\frac{k}{2^n}\right)+\frac{1+\tau}{2}F_n\left(\frac{k+1}{2^n}\right)\right)+F\left(\frac{1}{2}\right)\\
&=\frac{1-\tau}{1+\tau}F_{n+1}\left(\frac{2k+1}{2^{n+1}}\right)+F\left(\frac{1}{2}\right)\\
&=\frac{1-\tau}{1+\tau}F\left(\frac{2k+1}{2^{n+1}}\right)+F\left(\frac{1}{2}\right).
\end{aligned}\tag{4.7}
$$

Hence (4.4) is satisfied at $x=(2k+1)/2^{n+1}$, and hence by every member of $\mathscr{D}$. Next, we have

$$
\begin{aligned}
F\left(\frac{2k+1}{2^{n+1}}\right)&=F_{n+1}\left(\frac{2k+1}{2^{n+1}}\right)\\
&=\frac{1-\tau}{2}F_n\left(\frac{k}{2^n}\right)+\frac{1+\tau}{2}F_n\left(\frac{k+1}{2^n}\right)\qquad\text{(by (4.1))}\\
&=\frac{1-\tau}{2}F\left(\frac{k}{2^n}\right)+\frac{1+\tau}{2}F\left(\frac{k+1}{2^n}\right)\\
&=\frac{1-\tau}{2}\left(\frac{1+\tau}{2}F\left(\frac{k}{2^{n-1}}\right)\right)+\frac{1+\tau}{2}\left(\frac{1+\tau}{2}F\left(\frac{k+1}{2^{n-1}}\right)\right)\\
&\qquad\text{(by the induction hypothesis)}\\
&=\frac{1+\tau}{2}\left(\frac{1-\tau}{2}F\left(\frac{k}{2^{n-1}}\right)+\frac{1+\tau}{2}F\left(\frac{k+1}{2^{n-1}}\right)\right)\\
&=\frac{1+\tau}{2}\left(\frac{1-\tau}{2}F_{n-1}\left(\frac{k}{2^{n-1}}\right)+\frac{1+\tau}{2}F_{n-1}\left(\frac{k+1}{2^{n-1}}\right)\right)\\
&=\frac{1+\tau}{2}F_n\left(\frac{2k+1}{2^n}\right)\qquad\text{(by (4.1)),}\\
&=\frac{1+\tau}{2}F\left(\frac{2k+1}{2^n}\right).
\end{aligned}\tag{4.8}
$$

Hence (4.5) is satisfied also at $x = (2k+1)/2^{n+1}$, and thus at every member of $\mathscr{D}$. The lemma is proved. ∎

LEMMA 4.2. *The moments $c_n$ satisfy the recurrence relation*

$$c_n = \frac{1-\tau}{2(2^n-1)} \sum_{j=0}^{n-1} \binom{n}{j} c_j, \qquad n = 1, 2, \ldots. \tag{4.9}$$

*Proof.* The functional equations in Lemma 4.1 imply

$$\begin{aligned} &\text{(i)} \quad \frac{1-\tau}{1+\tau}\, dF(t) = dF\left(t+\frac{1}{2}\right), \qquad t \in \left[0, \frac{1}{2}\right], \\ &\text{(ii)} \quad dF\left(\frac{t}{2}\right) = \frac{1+\tau}{2}\, dF(t), \qquad t \in [0, 1]. \end{aligned} \tag{4.10}$$

We write

$$c_n = \int_0^1 t^n\, dF(t) = \int_0^{1/2} t^n\, dF(t) + \int_{1/2}^1 t^n\, dF(t) = I_1 + I_2, \tag{4.11}$$

$$\begin{aligned} I_1 &= \int_0^1 \left(\frac{t}{2}\right)^n dF\left(\frac{t}{2}\right) \\ &= 2^{-n}\left(\frac{1+\tau}{2}\right) \int_0^1 t^n\, dF(t) \qquad \text{(by (4.10))} \\ &= 2^{-n-1}(1+\tau)\, c_n. \end{aligned} \tag{4.12}$$

$$\begin{aligned} I_2 &= \int_{1/2}^1 t^n\, dF(t) = \int_0^{1/2} \left(t+\frac{1}{2}\right)^n dF\left(t+\frac{1}{2}\right) \\ &= \int_0^{1/2} \sum_{j=0}^n \binom{n}{j} 2^{-n+j} t^j\, dF\left(t+\frac{1}{2}\right) \\ &= \frac{1-\tau}{1+\tau} \sum_{j=0}^n \binom{n}{j} 2^{-n+j} \int_0^{1/2} t^j\, dF(t) \qquad \text{(use (4.10))} \\ &= \frac{1-\tau}{1+\tau} \sum_{j=0}^n \binom{n}{j} 2^{-n+j} \int_0^1 \left(\frac{t}{2}\right)^j dF\left(\frac{t}{2}\right) \\ &= \frac{1-\tau}{1+\tau} \sum_{j=0}^n \binom{n}{j} 2^{-n} \int_0^1 t^j \frac{(1+\tau)}{2}\, dF(t) \\ &= 2^{-n-1}(1-\tau) \sum_{j=0}^n \binom{n}{j} c_j. \end{aligned} \tag{4.13}$$

From (4.11) we see that

$$c_n = 2^{-n-1}(1+\tau)\, c_n + 2^{-n-1}(1-\tau) \sum_{j=0}^{n} \binom{n}{j} c_j, \tag{4.14}$$

and solving for $c_n$ gives the lemma. ▮

Recall that the exponential generating function of $c_n$ is $f(x) := \sum_{n=0}^{\infty} (c_n/n!)\, x^n$.

LEMMA 4.3. *$f(x)$ satisfies the functional equation*

$$f(x) = \left(\frac{(1-\tau)\, e^{x/2} + (1+\tau)}{2}\right) f\left(\frac{x}{2}\right), \tag{4.15}$$

*and iteration of this equation produces the infinite product representation*

$$f(x) = \prod_{k=1}^{\infty} \left(\frac{(1-\tau)\, e^{x/2^k} + (1+\tau)}{2}\right). \tag{4.16}$$

*Proof.* Since $f$ is entire, all the manipulations with power series to follow are valid. Taking a Cauchy product of series gives

$$e^x f(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad a_n = \sum_{j=0}^{n} \frac{c_j}{j!\,(n-j)!}. \tag{4.17}$$

Using the recurrence of Lemma 4.2 to express $a_n$ gives

$$\begin{aligned} a_n &= \frac{c_n}{n!} + \sum_{j=0}^{n-1} \frac{c_j}{j!\,(n-j)!} = \frac{c_n}{n!} + \frac{c_n}{n!} \frac{2(2^n-1)}{(1-\tau)} \\ &= -\frac{(1+\tau)}{(1-\tau)} \frac{c_n}{n!} + \frac{2}{(1-\tau)} \frac{c_n}{n!} 2^n. \end{aligned} \tag{4.18}$$

Thus

$$\begin{aligned} e^x f(x) &= \sum_{n=0}^{\infty} \left( -\frac{1+\tau}{1-\tau} \frac{c_n}{n!} + \frac{2}{1-\tau} \frac{c_n}{n!} 2^n \right) x^n \\ &= -\frac{1+\tau}{1-\tau} f(x) + \frac{2}{1-\tau} f(2x), \end{aligned} \tag{4.19}$$

or

$$f(2x) = \left(\frac{(1-\tau)\, e^x + (1+\tau)}{2}\right) f(x). \tag{4.20}$$

Letting $x \to x/2$ gives the first statement in the lemma. Since $f(\,)=1$, iteration yields the second statement. ∎

By the Cauchy integral theorem,

$$\frac{c_n}{n!} = \frac{1}{2\pi i} \oint_C \frac{f(t)}{t^{n+1}}\, dt, \tag{4.21}$$

where $C$ is a simple closed contour enclosing the origin. Let $g(x) := \ln f(x) - n \ln x$.

$$g(x) = \sum_{k=1}^{\infty} \ln\left(\frac{(1-\tau)\, e^{x/2^k} + (1+\tau)}{2}\right) - n \ln x. \tag{4.22}$$

Thus

$$g'(x) = \sum_{k=1}^{\infty} 2^{-k} \left(1 + \frac{1+\tau}{1-\tau} e^{-x/2^k}\right)^{-1} - \frac{n}{x}. \tag{4.23}$$

The critical points are the roots of the equation $g'(x) = 0$.

The analysis to follow closely parallels the analysis already done for the Cantor functions. In those cases where the proofs are essentially the same as those in the previous section, we omit the arguments.

LEMMA 4.4. *There is a unique positive root $\rho_n$ of $g'(x)=0$, and $\rho_n \to \infty$ as $n \to \infty$.*

*Proof.* See Lemma 3.2 of the previous section. ∎

Returning to the expression for $g'(x)$, (4.23), we see that

$$\sum_{k=1}^{\infty} \frac{1}{1 + ((1+\tau)/(1-\tau))\, e^{-\rho_n/2^k}} \frac{1}{2^k} = \frac{n}{\rho_n}. \tag{4.24}$$

Since $\rho_n \to \infty$ and

$$\left| \frac{1}{1 + ((1+\tau)/(1-\tau))\, e^{-\rho_n/2^k}} \frac{1}{2^k} \right| \leqslant \frac{1}{2^k}, \tag{4.25}$$

by the Lebesgue dominated convergence theorem we have

$$\frac{n}{\rho_n} \sim \sum_{k=1}^{\infty} \frac{1}{2^k} = 1, \tag{4.26}$$

i.e.,

$$\rho_n = n + d_n \qquad \text{with} \quad d_n = o(n). \tag{4.27}$$

We have to interrupt our proof with a

SUBLEMMA.

$$\sum_{k=1}^{\infty} \frac{e^{-n/2^k}}{1+((1+\tau)/(1-\tau))\, e^{-n/2^k}} \frac{1}{2^k} = \frac{L(n)}{n \ln 2} + o\left(\frac{1}{n}\right), \tag{4.28}$$

*where*

$$L(x) = \sum_{m=-\infty}^{\infty} E\left(1 - \frac{2m\pi i}{\ln 2}\right) x^{2m\pi i/\ln 2},$$

$$E(s) = \int_0^{\infty} \frac{x^{s-1}}{e^x + a}\, dx \qquad \textit{and} \qquad a := \frac{1+\tau}{1-\tau}.$$

*Proof.* The proof is quite similar to that of the sublemma of the Cantor case, but it has some individual features. Define

$$I(x) := \sum_{k=1}^{\infty} \frac{e^{-x/2^k}}{1+((1+\tau)/(1-\tau))\, e^{-x/2^k}} \frac{1}{2^k}. \tag{4.29}$$

We easily find the functional equation

$$2I(2x) = \frac{e^{-x}}{1+((1+\tau)/(1-\tau))\, e^{-x}} + I(x). \tag{4.30}$$

The Mellin transform of $I(x)$ exists for $0 < \sigma = \operatorname{Re}(s) < 1$. We have

$$\begin{aligned} \hat{I}(x) &= \int_0^{\infty} I(x)\, x^{s-1}\, dx = \frac{1}{2^{-s+1}-1} E(s), \\ E(s) &= \int_0^{\infty} \frac{e^{-x} x^{s-1}}{1+((1+\tau)/(1-\tau))\, e^{-x}}\, dx = \int_0^{\infty} \frac{x^{s-1}}{e^x + ((1+\tau)/(1-\tau))}\, dx. \end{aligned} \tag{4.31}$$

In the Cantor case, the corresponding $E(s)$ was expressed in terms of $\Gamma(s)$ and $\zeta(s)$. In that case, the use of familiar properties of these two functions made the analysis straightforward. For this case, however, we need to develop ad hoc properties of $E(s)$ which will enable us to use the Mellin inversion formula. We note that $E(s)$ is analytic in $\operatorname{Re}(s) > 0$. Let $s = \sigma + i\lambda$, $\sigma > 0$, $a = (1+\tau)/(1-\tau)$, $0 < \tau < 1$.

$$\begin{aligned} E(s) &= \int_0^{\infty} \frac{x^{s-1}}{e^x + a}\, dx = \int_{-\infty}^{\infty} \frac{e^{(\sigma + i\lambda)u}}{e^{e^u + a}}\, du \\ &= \int_{-\infty}^{\infty} \tilde{E}(u)\, e^{i\lambda u}\, du, \qquad \tilde{E}(u) := \frac{e^{\sigma u}}{e^{e^u} + a}. \end{aligned} \tag{4.32}$$

We take $u$ to be a complex variable. We integrate clockwise around a rectangle with vertices $(-R, R, R+i\delta, -R+i\delta)$. Note that the denominator of the integrand cannot vanish within the contour if $\delta$ is small enough, since the roots of $\exp(e^z)+a=0$ have imaginary parts $\arg(\ln a \pm i\pi)$. It is easily seen that the contribution over the vertical components of the contour goes to 0 as $R \to \infty$. Thus

$$E(s)=\int_{-\infty}^{\infty} \frac{e^{\sigma\xi + i\sigma\delta + i\xi\lambda - \lambda\delta}}{e^{e^{\xi}(\cos\delta + i\sin\delta)}+a}\,d\xi, \tag{4.33}$$

so

$$|E(s)| \leqslant e^{-\lambda\delta}\int_{-\infty}^{\infty} \frac{e^{\sigma\xi}}{|e^{e^{\xi}(\cos\delta + i\sin\delta)}+a|}\,d\xi,$$

so for $\lambda>0$ we see that there exists a positive number $\delta=\delta(a)$ such that

$$E(s)=O(e^{-\delta|\lambda|}), \qquad \lambda \to \pm\infty. \tag{4.34}$$

If $\lambda<0$, we shift the contour downward instead of upward. Note that the $O$-symbol depends on $a$, $\delta$, and $\sigma$, but not on $\lambda$.

The order estimate on $E(s)$ enables us to use the Mellin inversion formula. We have

$$I(x)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{E(s)}{2^{-s+1}-1}\,x^{-s}\,ds, \qquad 0<\sigma<1. \tag{4.35}$$

Shifting the contour to the right of 1 we pick up the residues at $s=1-2m\pi i/\ln 2$, $m \in \mathbf{Z}$. Thus

$$\begin{aligned} I(x) &= \frac{L(x)}{x\ln 2}+\frac{1}{2\pi i}\int_{\sigma'-i\infty}^{\sigma'+i\infty} \frac{E(s)}{2^{-s+1}-1}\,x^{-s}\,ds, \qquad \sigma'>1 \\ &= \frac{L(x)}{x\ln 2}+o\left(\frac{1}{x}\right). \end{aligned} \tag{4.36}$$

The proof of the sublemma is completed by noting that

$$I(n)=\sum_{k=1}^{\infty} \frac{e^{-n/2^k}}{1+((1+\tau)/(1-\tau))\,e^{-n/2^k}}\,\frac{1}{2^k}. \quad \blacksquare \tag{4.37}$$

Returning to (4.24), as in Section 3 (see, e.g., (3.29)) we still have

$$\sum_{k=1}^{\infty} \frac{1}{1+((1+\tau)/(1-\tau))\, e^{-\rho_n/2^k}} \frac{1}{2^k}$$
$$= \sum_{k=1}^{\infty} \frac{1}{1+((1+\tau)/(1-\tau))\, e^{-n/2^k}} \frac{1}{2^k} + o(1/n).$$

Thus

$$(n+d_n)\left(\sum_{k=1}^{\infty} \frac{1}{1+ae^{-n/2^k}} \frac{1}{2^k} + o\left(\frac{1}{n}\right)\right) = n, \quad (4.38)$$

$$n\left(1 - a\sum_{k=1}^{\infty} \frac{e^{-n/2^k}}{1+ae^{-n/2^k}} \frac{1}{2^k} + o\left(\frac{1}{n}\right)\right) + d_n\left(1 + O\left(\frac{1}{n}\right)\right) = n, \quad (4.39)$$

and using the sublemma gives

$$n\left(1 - \frac{aL(n)}{n \ln 2} + o\left(\frac{1}{n}\right)\right) + d_n + o(d_n) = n, \quad (4.40)$$

or

$$d_n = \frac{aL(n)}{\ln 2} + o(1). \quad (4.41)$$

We have just shown

LEMMA 4.5.

$$\rho_n = n + \frac{aL(n)}{\ln 2} + o(1). \quad \blacksquare \quad (4.42)$$

We write the Cauchy integral formula

$$\frac{c_n}{n!} = \frac{1}{2\pi i \rho_n^n} \oint_c \frac{f(\rho_n x)}{x^{n+1}}\, dx. \quad (4.43)$$

In much the same way in which the proof was done for the Cantor function we can prove

LEMMA 4.6.

$$\frac{1}{2\pi i} \oint \frac{f(\rho_n x)}{x^{n+1}}\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{f(\rho_n(1+iy))}{(1+iy)^{n+1}}\, dy. \quad \blacksquare \quad (4.44)$$

Now

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f(\rho_n(1+iy))}{(1+iy)^{n+1}}\,dy$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(\ln f(\rho_n(1+iy))-n\ln(1+iy))\frac{dy}{(1+iy)}. \tag{4.45}$$

Let

$$J_n(y)=\ln f(\rho_n(1+iy))=\sum_{k=1}^{\infty}\ln\left(\frac{(1-\tau)\,e^{x/2^k}+(1+\tau)}{2}\right). \tag{4.46}$$

We split the integral into two parts,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f(\rho_n(1+iy))}{(1+iy)^{n+1}}\,dy$$
$$=\frac{1}{2\pi}\int_{|y|\leqslant n^{-\varepsilon}}\exp(\ln f(\rho_n(1+iy))-n\ln(1+iy))\frac{dy}{(1+iy)}$$
$$+\frac{1}{2\pi}\int_{|y|\geqslant n^{-\varepsilon}}\frac{f(\rho_n(1+iy))}{(1+iy)^{n+1}}\,dy=S_1+S_2,\qquad \frac{1}{3}<\varepsilon<\frac{1}{2}. \tag{4.47}$$

LEMMA 4.7.

$$S_1\sim\frac{f(\rho_n)}{\sqrt{2\pi n}}. \tag{4.48}$$

*Proof.* The proof is nearly identical to that in the Cantor case, and we omit it. ∎

The result below differs slightly from that for the Cantor case.

LEMMA 4.8.

$$f(\rho_n)\sim e^{\rho_n}\rho_n^{-\ln(1+a)/\ln 2}\exp\left(\frac{\ln(1+a)}{2}+\frac{aE'(1)}{\ln 2}\right)e^{-aM(\rho_n)}, \tag{4.49}$$

*where*

$$M(x)=\frac{1}{2\pi i}\sum_{m=-\infty}^{\infty}{}'\,\frac{E(1-2m\pi i/\ln 2)}{m}\,x^{2m\pi i/\ln 2}.$$

*Remark.* $M(x)$ is real for $x>0$ and is periodic in the variable $\ln x$.

*Proof.* A little algebra shows that we have

$$\frac{d}{dx}\ln f(x)=1-\frac{1+\tau}{1-\tau}\sum_{k=1}^{\infty}\frac{e^{-x/2^k}}{1+ae^{-x/2^k}}\frac{1}{2^k}$$

$$=1-\frac{a}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{E(s)}{2^{-s+1}-1}x^{-s}\,ds,\qquad 0<\sigma<1,\quad (4.50)$$

where $E(s)$ is as in (4.32). We now choose a small positive number $b>0$ and integrate with respect to $x$ from $b$ to $x$.

$$\ln f(x)-\ln f(b)=x-b-\frac{a}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{E(s)}{2^{-s+1}}\frac{x^{-s+1}-b^{-s+1}}{-s+1}\,ds.\quad (4.51)$$

Since $0<\sigma<1$, $\operatorname{Re}(-s+1)>0$. Therefore in the limit as $b\to 0^+$ we have

$$\ln f(x)=x-\frac{a}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{E(s)}{2^{-s+1}-1}\frac{x^{-s+1}}{-s+1}\,ds,\qquad 0<\sigma<1.\quad (4.52)$$

We now shift the vertical contour to the right of 1, and at 1 we encounter a pole of order two and a simple pole at $s=1-2m\pi i/\ln 2$ for each nonzero $m\in\mathbf{Z}$. The residue at $s=1$ is

$$\operatorname{Res}(s=1)=\frac{-\ln(1+a)\ln x}{a\ln 2}+\left(\frac{\ln(1+a)}{2a}+\frac{E'(1)}{\ln 2}\right),\quad (4.53)$$

and for $m\neq 0$

$$\operatorname{Res}\left(s=1-\frac{2m\pi i}{\ln 2}\right)=-\frac{E(1-2m\pi i/\ln 2)}{2\pi i m}x^{2m\pi i/\ln 2}.$$

Therefore

$$\ln f(x)=x-a\left[-\operatorname{Res}(s=1)+M(x)+\frac{1}{2\pi i}\int_{\sigma'-i\infty}^{\sigma'+i\infty}(*)\,ds\right],\qquad \sigma'>1$$

$$=x-\frac{\ln(1+a)\ln x}{\ln 2}+\left(\frac{\ln(1+a)}{2}+\frac{aE'(1)}{\ln 2}\right)-aM(x)+O\left(\frac{1}{\chi^{\sigma'-1}}\right).$$
(4.54)

Thus

$$f(x)=e^{x}x^{-\ln(1+a)/\ln 2}\exp\left(\frac{\ln(1+a)}{2}+\frac{aE'(1)}{\ln 2}\right)e^{-aM(x)}\exp\left[O\left(\frac{1}{x^{\sigma'-1}}\right)\right], \tag{4.55}$$

and the indicated result follows. ▮

In a similar fashion we can show that $S_2=o(S_1)$. Consequently

$$\begin{aligned} c_n &\sim \frac{n!\,f(\rho_n)}{\rho_n^n\sqrt{2\pi n}}, \\ &\sim \exp\left(\frac{\ln(1+a)}{2}+\frac{aE'(1)}{\ln 2}\right)e^{-aM(\rho_n)}n^{-\ln(1+a)/\ln 2}. \end{aligned} \tag{4.56}$$

Similar to the case of the Cantor function we still have $M(\rho_n)=M(n)+O(1/n)$. A little bit of algebra gives the following surprising result for the asymptotics of the moments of the Riesz–Nagy function:

THEOREM 2.

$$\begin{aligned} c_n &\sim \left(\frac{2}{1-\tau}\right)^{1/2}\exp\left(\frac{aE'(1)}{\ln 2}\right)e^{-aM(n)} \\ &\cdot n^{-\ln(1+a)/\ln 2}, \qquad a=\frac{1+\tau}{1-\tau},\ E'(1)=\int_0^\infty \frac{\ln x}{e^x+a}\,dx. \quad ▮ \end{aligned} \tag{4.57}$$

Note that as $\tau\to 1^-$, $a\to\infty$ and $a>1$ for all $\tau$, $0<\tau<1$.

## 5. NUMERICAL RESULTS

It is interesting to compare the numerical values of $c_n$ from the recurrence relation in (3.4) with those in (3.82).

| $n$ | $c_n$ | Right-hand side of (3.82) |
|---|---|---|
| 5 | 0.242188 | 0.265840 |
| 10 | 0.163236 | 0.171669 |
| 20 | 0.108161 | 0.110857 |
| 50 | 0.061541 | 0.062186 |
| 100 | 0.039946 | 0.040157 |
| 150 | 0.030980 | 0.031093 |
| 200 | 0.025875 | 0.025932 |

Since $\alpha(t)$ is symmetric about $\frac{1}{2}$, the recursion relation (1.6) becomes

$$p_{n+1}(t)=(t-\tfrac{1}{2})\,p_n(t)-C_n\,p_{n-1}(t). \tag{5.1}$$

Below we give some tabular values of $C_n$, computed from Eqs. (1.5) and (1.6).

| $n$ | $C_n$ |
|---|---|
| 2 | 0.050000 000000 000000 |
| 3 | 0.091483 516483 516484 |
| 4 | 0.027682 600853 332560 |
| 5 | 0.126757 610894 292358 |
| 6 | 0.027839 060601 338983 |
| 7 | 0.081218 046335 405992 |
| 8 | 0.032401 605452 692483 |
| 9 | 0.151074 967691 985217 |
| 10 | 0.031094 180111 007488 |
| 11 | 0.089812 239727 242084 |
| 12 | 0.044544 361600 397104 |
| 13 | 0.102672 745477 775883 |
| 14 | 0.018920 106686 776286 |
| 15 | 0.141018 772732 868884 |
| 16 | 0.038697 778693 426457 |
| 17 | 0.080131 958556 034986 |
| 18 | 0.008672 273693 721123 |

The behavior of $C_n$ may have very likely a quasi-periodic structure in their index $n$ as pointed out to us by a referee (see [1, 3].) Recall the Rakhmanov result (1.7) which for certain $\alpha(t)$ gives $\lim_{n\to\infty} C_n = \frac{1}{16}$. It is pretty clear that the limit does not hold for the Cantor function. However, the sequence of arithmetic means that

$$\hat{C}_n := \frac{1}{n-1} \sum_{k=2}^{n} C_k \tag{5.2}$$

may very well converge to $\frac{1}{16}$, for instance, $\hat{C}_{18} = 0.0662.$[3]

[3] To obtain $C_n$ it was necessary to avoid a loss of significant figures in $G_n$. The loss is catastrophic. $G_n$ seems to have a magnitude about $10^{-k}$, where $k$ is $n(n-2)$. Since the moments $c_n$ decay slowly, this exponent represents roughly the significant figures lost in a straightforward evaluation of the determinants. The loss was avoided by doing all the operations in integer arithmetic using MAPLE, including the computation of the moments $c_n$. (This required manipulating rational numbers whose numerators and denominators were 1000-digit integers). Only at the last step were the figures converted to decimal representation. The values given are accurate to all places. We have given more places for these quantities than we gave for the moments to facilitate the application of summability methods to the sequence, should anyone care to do so. These remarks also apply to similar computations for the Riesz–Nagy functions.

We have also appended a short table of the moments $c_n$ of the Riesz–Nagy singular function for $\tau = \frac{1}{2}$ and compared them with the asymptotic lead term. Here $a = 3$. Note that $E'(1) = -0.062578739$.

| $n$ | $c_n$ | Right-hand side of (4.57) |
|---|---|---|
| 5 | 0.036398 | 0.061019 |
| 10 | 0.011551 | 0.015255 |
| 50 | 0.575518(−3) | 0.610187(−3) |
| 100 | 0.145195(−3) | 0.149541(−3) |
| 150 | 0.664655(−4) | 0.677986(−4) |
| 200 | 0.375702(−4) | 0.381367(−4) |

The values of the coefficients $C_n$ in the recurrence relation (1.4) are the following:

| $n$ | $C_n$ |
|---|---|
| 2 | 0.054081 632653 061224 |
| 3 | 0.064037 140851 089735 |
| 4 | 0.055094 142962 727365 |
| 5 | 0.071043 071430 359994 |
| 6 | 0.050748 893696 793874 |
| 7 | 0.069932 783984 562577 |
| 8 | 0.063336 870947 548931 |
| 9 | 0.055038 382031 551573 |
| 10 | 0.061642 069230 379927 |
| 11 | 0.074309 849404 644070 |
| 12 | 0.052524 554646 968216 |
| 13 | 0.062715 799920 415958 |
| 14 | 0.060168 499052 997664 |
| 15 | 0.062912 947574 403845 |
| 16 | 0.071113 030506 618068 |
| 17 | 0.058758 087546 552993 |

Recall the Rakhmanov result: $\lim_{n \to \infty} C_n = \frac{1}{16}$ for appropriate functions. The values of $C_n$ seem to hover about $\frac{1}{16} = 0.0625$. Perhaps their behavior is also quasi periodic. The figures suggest a convergence of the sequence of arithmetic means

$$\hat{C}_n := \frac{1}{n-1} \sum_{k=2}^{n} C_k,$$

to $\frac{1}{16}$ more vigorous than that of the mean values of $C_n$ for the Cantor function. This is probably due to the fact that this function satisfies at least one of the Rakhmanov criteria: it is strictly increasing.

In the limit as $\tau \to 0$, $\alpha(t) \to t$, and we get the monic shifted Legendre polynomials, with $C_n = n^2/16(n^2 - \frac{1}{4})$. One would expect that for smaller values of $\tau$ the $C_n$ would hover even more closely around $\frac{1}{16}$. Furthermore, the sequence of arithmetic means seem to converge much more rapidly to $\frac{1}{16}$ than those for $\tau = \frac{1}{2}$.

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